Research Training Group – Uncertainty and Randomness in Algorithms, Verification, and Logic

Chair for Logic and Theory of Discrete Systems

Homomorphism Tensors and Linear Equations

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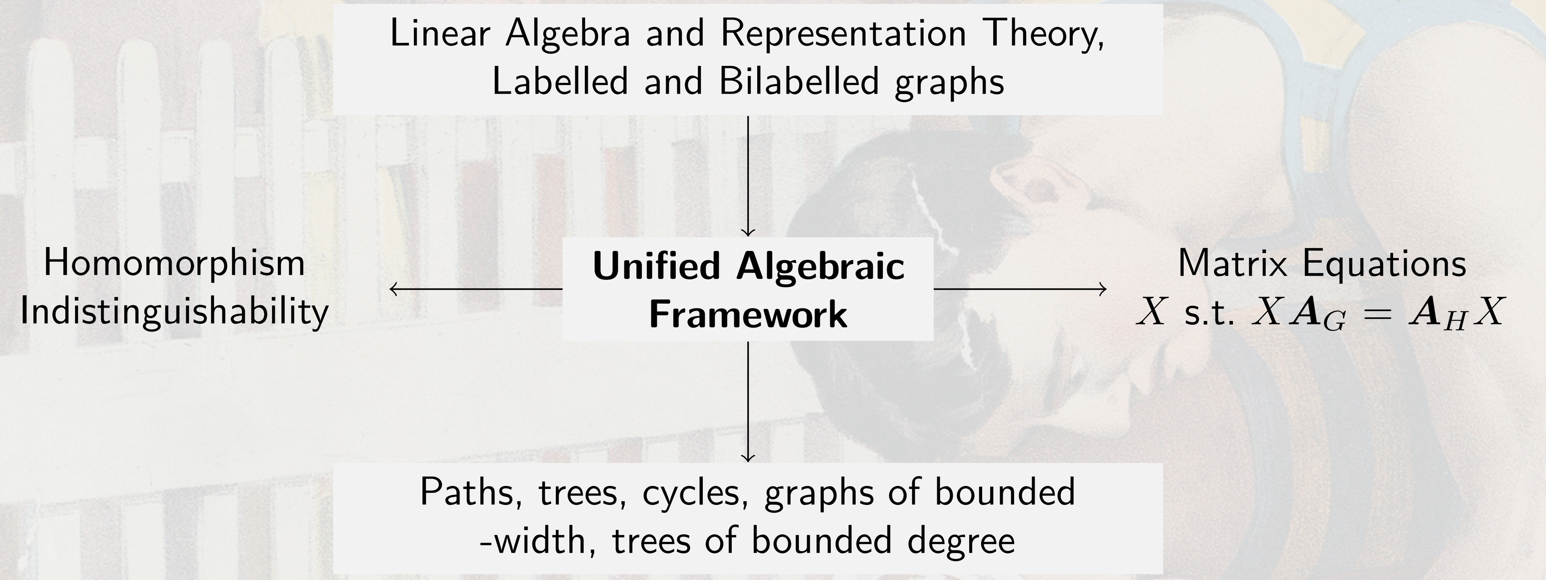
Motivation

Lovász (1967) proved that graphs G and H are isomorphic if and only if they are *homomorphism indistinguishable* over all graphs F the number of homomorphisms $F \to G$ equals the number of homomorphisms $F \to H$.

Homomorphism indistinguishability over restricted graph classes gives rise to a wide range of equivalence relations which can be characterised in terms of systems of equations. For example, graphs G and H are homomorphism indistinguishable over cycles/trees/path if and only if the system $X\boldsymbol{A}_G = \boldsymbol{A}_H X$ has an invertible/doubly-stochastic/pseudo-stochastic solution $X \in \mathbb{C}^{V(H)\times V(G)}$. We set out to provide a uniform explanation for such results.



Labelled Graphs and Homomorphism Tensors

A *labelled graph* $\boldsymbol{F}$ is a tuple of a graph F and a vertex $u \in V(F)$. Given a graph G, the *homomorphism tensor* of $\boldsymbol{F}$ is $\boldsymbol{F}_G \in \mathbb{C}^{V(G)}$ where

$$\boldsymbol{F}_G(v) := \text{number of homomorphisms } h\colon F \to G \text{ such that } h(u) = v$$

for all $v \in V(G)$. This can be extended to *bilabelled graphs* $\boldsymbol{F} = (F, u_1, u_2)$ which carry an *in-label* $u_1 \in V(F)$ and an *out-label* $u_2 \in V(F)$. Their homomorphism tensors $\boldsymbol{F}_G$ represent matrices in $\mathbb{C}^{V(G)\times V(G)}$.

Example For every graph G, the homomorphism tensor $\boldsymbol{A}_G$ of the bilabelled graph $\boldsymbol{A} = \overset{1\ 2}{\bullet\!\!-\!\!\bullet}$ is the adjacency matrix of G.

Operations

Combinatorial operations on (bi)labelled graphs correspond to algebraic operations on homomorphism tensors.

- The *sum-of-entries* $\operatorname{soe} \boldsymbol{F}_G$ equals $\hom(F, G)$, the homomorphism count of the *underlying unlabelled* graph F of $\boldsymbol{F}$.
- The *matrix product* $\boldsymbol{F}_G \cdot \boldsymbol{F}'_G$ equals the homomorphism matrix of the bilabelled graph obtained from $\boldsymbol{F}$ and $\boldsymbol{F}'$ by *series composition*.
- The *Schur product* $\boldsymbol{F}_G \odot \boldsymbol{F}'_G$ equals the homomorphism vector of the labelled graph obtained from $\boldsymbol{F}$ an $\boldsymbol{F}'$ by *gluing*.

Example The bilabelled graph $\overset{1\quad\ 2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}$ results from the series composition $\overset{1\ 2}{\bullet\!\!-\!\!\bullet} \cdot \overset{1\ 2}{\bullet\!\!-\!\!\bullet}$. Its homomorphism matrix is $\boldsymbol{A}_G^2 = \boldsymbol{A}_G \cdot \boldsymbol{A}_G$.

Inner-Product Compatible Graph Classes

Using linear algebra, we obtain matrix equations for homomorphism indistinguishability over classes of labelled graphs $\mathcal{R}$ which are

- *inner-product compatible*, i.e. for all $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{R}$ the homomorphism counts from the graph obtained by gluing $\boldsymbol{R}$ and $\boldsymbol{S}$ and forgetting labels, are equal to the homomorphism counts from some graph in $\mathcal{R}$, and
- *$\boldsymbol{A}$-invariant*, i.e. for every labelled graph $\boldsymbol{R} = (R, u) \in \mathcal{R}$, the labelled graph $\boldsymbol{A} \cdot \boldsymbol{R}$ obtained by adding a fresh vertex u' to R, adding the edge uu', and placing the label on u', is also in $\mathcal{R}$.

Example The family of labelled paths with labels at end vertices is inner-product compatible. For example,

$$\operatorname{soe}\left(\overset{1}{\bullet\!\!-\!\!\bullet} \odot \overset{1}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}\right) = \operatorname{soe}\left(\bullet\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\bullet\!\!-\!\!\bullet\right) = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet = \operatorname{soe}\left(\overset{1}{\bullet}\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\right)$$

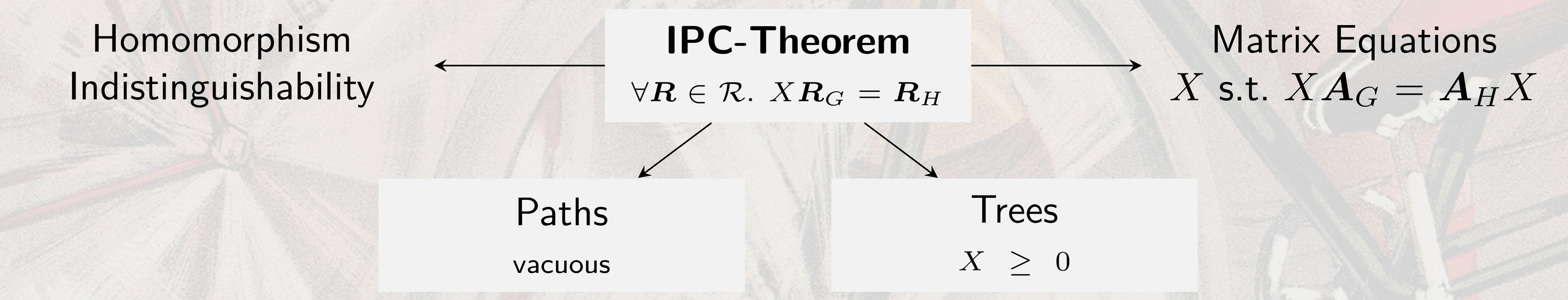
It is also $\boldsymbol{A}$-invariant. For example, $\boldsymbol{A} \cdot \overset{1}{\bullet\!\!-\!\!\bullet} = \overset{1\ 2}{\bullet\!\!-\!\!\bullet} \cdot \overset{1}{\bullet\!\!-\!\!\bullet} = \overset{1}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}$.

IPC-Theorem Let $\mathcal{R}$ be an inner-product compatible and $\boldsymbol{A}$-invariant family of labelled graphs containing $\overset{1}{\bullet}$. Then for graphs G and H the following are equivalent:

1. G and H are homomorphism indistinguishable over $\mathcal{R}$,
2. There exists a pseudo-stochastic $X \in \mathbb{Q}^{V(H)\times V(G)}$ such that $X\boldsymbol{R}_G = \boldsymbol{R}_H$ for all $\boldsymbol{R} \in \mathcal{R}$.

Trees and Paths

We apply our theorem to the classes of trees and paths and prove known characterisation of homomorphism indistinguishability over these classes in a uniform manner. In particular, we find a combinatorial explanation for the obscurity that these characterisations differ only in the constraint $X \geq 0$.



Trees of Bounded Degree

Characterising homomorphism indistinguishability over graph classes of bounded degree is a notoriously difficult problem. For trees of bounded degree, we prove the following.

Theorem For every $d \in \mathbb{N}$, there exist graphs G and H such that

- G and H are homomorphism indistinguishable over trees of degree $\leq d$ and
- G and H are not homomorphism indistinguishable over all trees.

In particular, it is not possible to simulate the 1-dimensional Weisfeiler–Leman algorithm (Colour Refinement) by counting homomorphisms from trees of bounded degree.

Specht–Wiegmann Theorem

We use representation theory to derive novel matrix equations characterising homomorphism indistinguishability. The recipe is the following:

1. Definition of an involution monoid, for example the *path monoid* $\mathcal{P} = \{\overset{1,2}{\bullet}, \overset{1\ 2}{\bullet\!\!-\!\!\bullet}, \overset{1\quad\ 2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}, \overset{1\qquad\ 2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}, \dots\}$.
2. For a graph G, define a representation $\mathcal{P} \to \mathbb{C}^{V(G)\times V(G)}$ mapping $\boldsymbol{P}$ to its homomorphism tensor $\boldsymbol{P}_G$.
3. The sum-of-entries of this representation counts the homomorphisms of interests. It can be interpreted as a character of a certain subrepresentation. The desired matrix equation arises from the following theorem:

Theorem Let $\varphi\colon \Gamma \to \mathbb{C}^{V\times V}$ and $\psi\colon \Gamma \to \mathbb{C}^{W\times W}$ be finite-dimensional representations of an involution monoid Γ. Then the following are equivalent:

1. For all $g \in \Gamma$, $\operatorname{soe} \psi(g) = \operatorname{soe} \varphi(g)$.
2. There exists a pseudo-stochastic $X \in \mathbb{C}^{W\times V}$ such that $X\varphi(g) = \psi(g)X$.

Graphs of Bounded Pathwidth

Extending the known characterisation of homomorphism indistinguishability over graphs of treewidth $\leq k$ in terms of the existence of a *non-negative* solution to the Sherali–Adams-style relaxation $\mathsf{L}^{k+1}_{\text{iso}}(G, H)$ of the ILP for graph isomorphism, we prove the following:

Theorem Let $k \in \mathbb{N}$. Graphs G and H are homomorphism indistinguishable over graphs of pathwidth $\leq k$ if and only if $\mathsf{L}^{k+1}_{\text{iso}}(G, H)$ has a *rational* solution.

Graphs of Bounded Treedepth

Our techniques yield a novel system of equations characterising homomorphism indistinguishable over graphs of bounded treedepth.

Theorem Let $k \in \mathbb{N}$. Graphs G and H are homomorphism indistinguishable over graphs of treedepth $\leq k$ if and only if the system of equations stated below has a rational solution.

$$\sum_{v' \in V(G)} X(\boldsymbol{w}w, \boldsymbol{v}v') = X(\boldsymbol{w}, \boldsymbol{v}) \quad \text{for all } w \in V(H) \text{ and } \boldsymbol{v} \in V(G)^\ell,\ \boldsymbol{w} \in V(H)^\ell \text{ where } 0 \leq \ell < k.$$

$$\sum_{w' \in V(H)} X(\boldsymbol{w}w', \boldsymbol{v}v) = X(\boldsymbol{w}, \boldsymbol{v}) \quad \text{for all } v \in V(G) \text{ and } \boldsymbol{v} \in V(G)^\ell,\ \boldsymbol{w} \in V(H)^\ell \text{ where } 0 \leq \ell < k.$$

$$X(\boldsymbol{w}, \boldsymbol{v}) = 0 \quad \text{if not } \boldsymbol{v}_i = \boldsymbol{v}_{i+1} \iff \boldsymbol{w}_i = \boldsymbol{w}_{i+1} \text{ for all } i < k.$$

$$X((), ()) = 1 \quad \text{for the empty tuple } ().$$

References

[1] M. Grohe, G. Rattan and T. Seppelt. 'Homomorphism Tensors and Linear Equations'. In: *49th International Colloquium on Automata, Languages, and Programming, ICALP 2022, July 4-8, 2022, Paris, France*. Ed. by M. Bojańczyk, E. Merelli and D. P. Woodruff. Vol. 229. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi: `10.4230/LIPIcs.ICALP.2022.26`.